Applied Math Prelim Exam - August 10, 2022

Justify all your answers.

1.) **a)** Compute the fundamental solution of the initial value problem for the heat equation,

$$\begin{cases} \frac{\partial G}{\partial t} - \Delta G = 0 & (x,t) \in \mathbb{R}^d \times (0,\infty)\,, \\ G(x,0) = \delta_0(x) & x \in \mathbb{R}^d\,, \end{cases}$$

b) Write a representation formula to the solution of the initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = 0 & (x,t) \in \mathbb{R}^d \times (0,\infty)\,, \\ u(x,0) = f(x) & x \in \mathbb{R}^d\,, \end{cases}$$

i) - Show that this formula makes sense for $f \in \mathcal{S}'$ (i.e. f is a tempered distribution).

ii) - Show that the solution $u(x,t)$ of this problem is in $C^\infty(\mathbb{R}^d \times (0,\infty))$ for any initial data in $\mathcal{S}'$.

2.) Let the bounded domain $\Omega \subset \mathbb{R}^d$ to have a smooth boundary. For a given $f \in H^s(\Omega)$, consider the problem of finding $u = u(x)$ solution to the boundary value problem

$$\Delta u - K\,u = f, \qquad \text{in } \Omega, \quad \text{for } K \geq 0,$$
$$u = u_D(x) \qquad \text{for } x \in \partial\Omega.$$

a) In order for the boundary value problem to have a unique solution $u \in H^{s+2}(\Omega)$,
i.) How regular does $\partial\Omega$ need to be?
ii.) For any $x \in \partial\Omega$, on which Sobolev space must the boundary data $u = u_D(x)$ belong to?

b) Write the estimates for the solution $u(x) \in H^{s+2}(\Omega)$ showing the dependence with respect to data f, u_D, Ω, and K in terms of their Sobolev norms.

c) For what values of Sobolev exponent s will the solution $u(x)$ be continuous?

3.) Use the Contraction Mapping Theorem to prove local existence and uniqueness for the initial-value problem

$$\begin{cases} \frac{dq}{dt} = q^2 + t, & t \in (0,T), \\ q(0) = 1. \end{cases}$$

Give a lower bound for T, the length of the time interval for which the solution is guaranteed to exist.