The following three problems are weighted equally; two complete solutions (or a complete solution and two half-solutions) are required for a passing grade. A correct partial solution (in which it is clear exactly what was proven) is preferrable to a claimed full solution with errors. In Problem 3, each computation must be accompanied by a justification of the computation's correctness.

Throughout this exam, we say that the random variable ξ is a *coin flip* if

$$P(\xi = 1) = P(\xi = -1) = \frac{1}{2}.$$

Problem 1

i) Show that for any random variable X, and any $s, t \geq 0$,

$$P(X \geq t) \leq e^{-st} E(e^{sX}).$$

ii) Let $\xi_1, \ldots, \xi_n$ be independent coin flips and let $X_n = \sum_{i=1}^n \xi_n$. Prove that for any $t \geq 0$,

$$P(X_n \geq t\sqrt{n}) \leq e^{-t^2/2}.$$

Problem 2

For random variables X and Y, define

$$d(X, Y) = \inf\{\epsilon \geq 0 : P(|X - Y| > \epsilon) \leq \epsilon\}.$$

Prove that d metrizes convergence in probability, in the sense that $X_n \to X$ in probability if and only if $d(X_n, X) \to 0$.

Problem 3

Let $\xi_1, \xi_2, \ldots$ be i.i.d. coin flips. Let $X_n = \sum_{i=1}^n \xi_n$, and let

$$T = \inf\{n \geq 4 : \xi_n = -1 \text{ and } \xi_{n-1} = \xi_{n-3} = 1\}.$$

i) Compute $E(X_T)$.

ii) Compute $E(X_{T+1})$.

iii) Compute $E(X_{T-1})$.