PRELIMINARY EXAMINATION IN TOPOLOGY: PART 2

August 2022 2 Hours

Work all 3 problems. Explain your work carefully. The problems are weighted equally.

1. State clearly whether each of the following unrelated assertions is true or false. Prove your answer with a deductive argument, an example, a counterexample, etc.
   (a) If $f, g\colon X \to X$ are two smooth maps on a manifold X, and the Lefschetz numbers satisfy $L(f) = L(g)$, then f is homotopic to g.
   (b) There exists a degree one map $f\colon S^1 \times S^1 \times S^1 \to S^3$.
   (c) There exists a smooth nonzero 4-form on $\mathbb{RP}^4$.

2. Let X be a smooth manifold of dimension at least 1. Let ω_x be a nonzero element of T_x^*X for some $x \in X$.
   (a) Construct a smooth 1-form ω on X whose value at x is ω_x.
   (b) Can we always construct ω in part (a) to be *closed*? Proof or counterexample.

3. Consider the equation
$$(x^1)^2 + (x^2)^2 = (x^3)^2 + (x^4)^2.$$
   Interpret (x^1, x^2, x^3, x^4) as a vector in $\mathbb{R}^4$.
   (a) Explain how the equation defines a subset S of $\mathbb{RP}^3$.
   (b) Prove that S is a submanifold of $\mathbb{RP}^3$.
   (c) Find a familiar manifold M diffeomorphic to S, construct a diffeomorphism $M \to S$, and prove that your map is a diffeomorphism.