PRELIMINARY EXAMINATION IN ALGEBRA
PART I
JANUARY 15, 2021

Please solve at least 3 of the following 4 problems.

(1) A group is called an elementary abelian p-group if it is isomorphic to $(\mathbb{Z}/p)^n$ for some n. Suppose G is a solvable finite group. Prove that it has an elementary abelian subgroup A which is characteristic in G, i.e., $\sigma(A) = A$ for all $\sigma \in \mathrm{Aut}(G)$.

Hint: if H is any abelian group then the map $x \mapsto x^p$ is a homomorphism from H to itself.

(2) Show that if $|G| = 132 = 2^2 \cdot 3 \cdot 11$ then G is not simple.

(3) Let $R = \mathbb{Z}[\sqrt{10}]$. Note

$$6 = 2 \cdot 3 = (4 + \sqrt{10})(4 - \sqrt{10}).$$

Are 2 or 3 irreducible in R? Are they prime in R? Is R a UFD? (Prove your answers.)

(4) Prove that the rings $F[x, y]/(y^2 - x)$ and $F[x, y]/(x^2 - y^2)$ are non-isomorphic for any field F.