Real Analysis January 2021 Prelim Exam

Problem 1: Prove the following theorem: *Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$ and $\{f_j\}_j$ $\mathcal{M}$-measurable functions with $f_j(x) \to f(x)$ for μ-a.e. $x \in X$. For every $\varepsilon > 0$ there exists X_ε such that $\mu(X_\varepsilon) < \varepsilon$ and $f_j \to f$ uniformly on $X \setminus X_\varepsilon$.*

Problem 2: Show that the conclusion of the previous theorem may fail when $\mu(X) = +\infty$.

Problem 3: Let $f \in L^1(X, \mu)$. Show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $E \in \mathcal{M}$ with $\mu(E) < \delta$, then

$$\int_E |f| \, d\mu < \varepsilon \, .$$

Then, find (X, μ) and a sequence $\{f_j\}_j \subset L^1(X, \mu)$ for which there exists $\varepsilon > 0$ with the property that for every $\delta > 0$

$$\sup_{E : \mu(E) < \delta} \sup_j \int_E |f_j| \, d\mu \geq \varepsilon \, .$$

Problem 4: Show that if X is a complete metric space and X is the countable union of closed subsets X_j, then at least one of the X_j has non-empty interior.

Problem 5: Give an example of a sequence which is weakly converging in $L^2(\mathbb{R})$ but does not admit any subsequence that is converging pointwise on $\mathbb{R}$.