Applied Mathematics Preliminary Exam, Part A

January 11, 2021

Work 3 *of the following* 4 *problems.*

1. Let $A : X \to Y$ and $B : Y \to Z$ be linear maps, where X, Y, Z are given Banach spaces. Assume that B and BA are bounded. If B is one-to-one, show that A is bounded.

2. Let X and Y be Banach spaces. Let $n \mapsto A_n$ be a sequence of bounded linear operators from X to Y, such that $n \mapsto A_n x$ converges for all x in some dense subset of X. Prove that $n \mapsto A_n x$ converges for all $x \in X$ if and only if $\sup_n \|A_n\| < \infty$.

3. Let $A : X \to Y$ be a linear operator defined on a dense subspace X of a Banach space Y. Assume that A has an inverse that is compact as a linear operator on Y. Show that Y is separable, and that the spectrum of A consists of eigenvalues only.

4. Consider $\mathrm{L}^2 = \mathrm{L}^2(\mathbb{R})$. Given any real number s, define $T_s : \mathrm{L}^2 \to \mathrm{L}^2$ by setting $(T_s x)(t) = x(t+s)$ for every $x \in \mathrm{L}^2$ and $t \in \mathbb{R}$. For $s \neq 0$ define also $D_s = \frac{1}{2s}[T_s - T_{-s}]$. Show that $\exp(D_s)$ converges strongly on L^2 to T_1 as $s \to 0$.
(*Hint*. The Fourier transform can be useful here.)