PRELIMINARY EXAMINATION: APPLIED MATHEMATICS — Part II

January 11, 2021

Work all 3 of the following 3 problems.

1. Let the field be real and $\Omega \subset \mathbb{R}^d$ be a domain with a Lipschitz boundary. For $w \in L^\infty(\Omega)$, define

$$H_w(\Omega) = \big\{ f \in L^2(\Omega) : \nabla(wf) \in (L^2(\Omega))^d \big\}.$$

(**a**) Give reasonable conditions on w so that $H_w(\Omega) = H^1(\Omega)$.

(**b**) Prove that $H_w(\Omega)$ is a Hilbert space. What is the inner-product?

(**c**) Suppose that Ω is bounded. Prove that there is a constant $C > 0$ such that for all $f \in H_w(\Omega)$ satisfying $\int_\Omega w(x)\, f(x)\, dx = 0$,

$$\|f\|_{L^2(\Omega)} \le C\big\{ \|\nabla(wf)\|_{L^2(\Omega)} + \|(1-w)f\|_{L^2(\Omega)} \big\}.$$

[Hint: use the usual Poincaré inequality for functions with zero average value.]

2. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, $a \in (L^\infty(\Omega))^d$, and $f \in L^p(\Omega)$ (for some p). Consider the boundary value problem (BVP)

$$\begin{aligned} -\Delta u + a \cdot \nabla u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

(**a**) Pose the BVP as a variational problem (VP) in $H_0^1(\Omega) \times H_0^1(\Omega)$. [You do *not* need to justify the equivalence.]

(**b**) Use the Sobolev Embedding Theorem to find the range of $p \ge 1$ such that your VP is well posed.

(**c**) Determine a bound on $\|a\|_{(L^\infty(\Omega))^d}$ (which will depend on the Poincaré constant C_P in $\|v\|_{L^2(\Omega)} \le C_P \|\nabla v\|_{L^2(\Omega)}$) so that you have coercivity, and then apply the Lax-Milgram Theorem to show existence and uniqueness of a solution.

3. Let X be a Banach space and $g : X \to X$ be a nonlinear mapping that is C^1 and has $g(0) = 0$ and $Dg(0) = 0$. For $f \in X$, we want to solve

$$F(u) = u + g(u) = f.$$

We consider the map $G(u) = u + \alpha(F(u) - f)$ for some $\alpha \in \mathbb{R}$.

(**a**) Show that $G(u)$ is a contractive map for small enough u and properly chosen α.

(**b**) Use the Banach contraction mapping theorem to show that there is a solution to $F(u) = f$, provided f is sufficiently small.

(**c**) Solve $F(u) = f$ using the inverse function theorem, provided f is sufficiently small.