ALGEBRA PRELIMINARY EXAM: PART I

Problem 1

Let H be a simple group of order 60. Prove that $H \simeq \mathrm{A}_5$ and determine all p-Sylow subgroups of H up to isomorphism.

Hint: Let N is a subgroup of H with n distinct conjugates. The action of H on the conjugates of N induces a homomorphism $\phi_N : H \to \mathrm{S}_n$.

Problem 2

Let R be an integral domain containing a field k as a subring. Assume that R is a finite dimensional vector space over k under the ring multiplication. Show that R is a field.

Problem 3

Consider the ring $\mathbb{Z}[x]$ and its quotients $\mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{-13}]$.

i) Prove that $\mathbb{Z}[x]$ is not a principal ideal domain.

ii) Prove that $\mathbb{Z}[i]$ is a principal ideal domain and unique factorization domain. Determine its units.

iii) Prove that $\mathbb{Z}[\sqrt{-13}]$ is not a unique factorization domain and hence not a principal ideal domain. Give an explicit example of an ideal of $\mathbb{Z}[\sqrt{-13}]$ which is not principal (prove your claim).

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