ALGEBRA II QUALIFYING EXAM

AUGUST 20TH, 2021

Problem 1. Let $n \in \mathbb{Z}$ be an integer and let $f_n(t) = t^3 - t + n \in \mathbb{Q}[t]$.

(a) Suppose $3 \nmid n$. Show that f_n is irreducible.
(b) Suppose that f_n is irreducible. Show that its Galois group is the symmetric group S_3.
(c) What isomorphism classes of groups can arise as Galois groups of f_n (for f_n possibly not irreducible)? For each possibility, provide some value of n realizing the specific Galois group.

Problem 2. Suppose k is a field and $f \in k[t]$ is a degree n separable polynomial with splitting field K. Let $r_1, \ldots, r_n \in K$ be the roots of f.

(a) Show that K is generated (as a k-algebra) by $r_1, \ldots, r_{n-1}$.
(b) Suppose K/k has degree $n!$. Show that the subfield of K generated by $r_1, \ldots, r_{n-2}$ is properly contained in K.

Problem 3. Let p be an odd prime and let $\zeta_p \in \mathbb{C}$ denote a primitive pth root of unity.

There are unique integers $a_1, a_2, \ldots, a_{p-1}$ such that:

- $a_1 = 1$.
- For $G := \sum_{i=1}^{p-1} a_i \zeta_p^i$, $G \notin \mathbb{Q}$ but $G^2 \in \mathbb{Q}$.

Determine the values of a_i and G^2.

(Hint: for calculating G^2, it helps at one point to use the automorphism of $\mathbb{F}_p^\times \times \mathbb{F}_p^\times$ given by $(i, j) \mapsto (i, \frac{i}{j})$.)