PRELIMINARY EXAMINATION IN ANALYSIS

Part II, Complex Analysis

August 17, 2021

---

1. Consider the right half-plane $H = \{z \in \mathbb{C} : \text{Re}(z) > 0\}$. Find all analytic functions $f : H \to H$ that satisfy $f(2) = 1/2$ and $f(1/2) = 2$.

2. Let f be a meromorphic function on $\mathbb{C}$ that satisfies $f(z)f(-z) = 1$ for all $z \in \mathbb{C}$. Show that there exists an entire function g such that $f(z) = g(z)/g(-z)$ for all $z \in \mathbb{C}$. (For simplicity you may assume that $f(0) = 1$.)

3. Let f be an entire function satisfying a bound $|f(z)| \leq \exp(|z|^n)$ for all $z \in \mathbb{C}$, where n is some positive integer. If $f(z) = 0$ whenever $\exp(\exp z) = 1$, show that $f = 0$.

4. Let $f_1, f_2, f_3, \ldots$ be bounded analytic functions on $\Omega = \{z \in \mathbb{C} : |z| > 1\}$ that take values in Ω. If the sequence $n \mapsto f_n(k)$ converges for each integer $k > 1$, show that the sequence $n \mapsto f_n$ converges uniformly on compact subsets of Ω.