PRELIMINARY EXAMINATION: APPLIED MATHEMATICS — Part I

August 16, 2021

Work all 3 of the following 3 problems.

1. Let $c_0 \subset \ell^\infty$ be the set of complex valued sequences converging to 0, endowed with the ℓ^∞-norm. For any $y \in \ell^1$, we define $f_y : c_0 \to \mathbb{C}$ by

$$f_y(x) = \sum_{j=1}^{\infty} x_j \bar{y}_j.$$

(**a**) Show that for any $y \in \ell^1$, $f_y \in (c_0)'$, the dual of c_0, with $\|f_y\|_{(c_0)'} \leq \|y\|_{\ell^1}$.

(**b**) Show that $\|f_y\|_{(c_0)'} = \|y\|_{\ell^1}$. [Hint: Consider the sequence (x^n) of elements of c_0 defined as $x_j^n = 0$ for $j \geq n$, and x_j^n is equal to either 0 (if $y_j = 0$) or $y_j/|y_j|$ (if $y_j \neq 0$) for $j < n$.]

(**c**) Show that every $f \in (c_0)'$ is of the form f_y for some $y \in \ell^1$.

2. Let H be a real Hilbert space and suppose that P is a bounded linear projection on H. Let $Q = I - P$ and define $M = P(H)$ and $N = Q(H)$. Suppose that M and N are closed. Recall that $Px = x$ for all $x \in M$ and that $M \cap N = \{0\}$.

(**a**) Show that there exists $C > 0$ such that

$$\|x - Px\| \leq C \inf_{y \in M} \|x - y\| \quad \text{for all } x \in H.$$

[Hint: Relate this to the *orthogonal* projection $\mathcal{P}_M$.]

(**b**) Prove that P is an orthogonal projection if and only if

$$\inf_{\substack{y \in N, \|y\|=1 \\ x \in M}} \|y - x\| = 1.$$

[Hint: For the converse, it is enough to show that for any $z \in H$, $z - Pz = Qz \perp M$. Consider $y = Qz/\|Qz\|$.]

3. Let X be a Banach space with dual X^*. Let $\{L_n\}_{n=1}^\infty \subset X^*$ and $\{x_n\}_{n=1}^\infty \subset X$. Assume that $L_n \to L \in X^*$ in the weak-* sense, and $x_n \to x$ in the norm of X.

(**a**) State the Uniform Boundedness Principle.

(**b**) Show that if X is a reflexive Banach space, then $L_n(x_n) \to L(x)$.

(**c**) Give an example to show that if we replace strong convergence by weak convergence of x_n, so instead only $x_n \overset{w}{\rightharpoonup} x$ in X, then (**c**) does not hold. [Hint: Consider ℓ^2 and $x_n = e_n$, where $(e_n)_n = 1$ and $(e_n)_m = 0$ for $m \neq n$.]