# PRELIMINARY EXAMINATION:
# APPLIED MATHEMATICS — Part II

August 16, 2021

*Work all 3 of the following 3 problems.*

**1.** (Poincaré type inequalities.) Consider a connected, bounded domain $\Omega \subset \mathbb{R}^d$ with a smooth boundary.

(**a**) Let $u_n$ be a bounded sequence of $H^1(\Omega)$. Assume that $\nabla u_n$ converges to 0 in $L^2(\Omega)$. Show that $u_n$ converges strongly in $L^2(\Omega)$ to a constant $a$.

(**b**) In addition, assume that $\gamma_0(u_n)$, the trace of of $u_n$ on $\partial\Omega$, converges to 0 in $L^2(\partial\Omega)$. Show that $u_n$ converges to 0 in $L^2(\Omega)$.

(**c**) Show that there exists $C > 0$, depending only on $\Omega$, such that for any $u \in H^1(\Omega)$:

$$\|u\|^2_{L^2(\Omega)} \leq C(\|\nabla u\|^2_{L^2(\Omega)} + \|\gamma_0(u)\|^2_{L^2(\partial\Omega)}). \tag{1}$$

[Hint: You can give a proof by contradiction using (**a**) and (**b**).]

**2.** Let $\Omega$ be a connected, bounded domain with a smooth boundary, and consider the differential boundary value problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ -\nabla u \cdot \nu &= g + u && \text{on } \partial\Omega. \end{aligned}$$

(**a**) Develop an appropriate weak or variational form for the problem. In what Sobolev spaces should $f$ and $g$ lie?

(**b**) Prove that there exists a unique solution to the problem. [Hint: You may use the Poincaré inequality (1).]

**3.** Let $X$ be a Banach space and $F : X \to X$ be a smooth map. Suppose that $x_*$ is a simple root of $F$ in the sense that $F(x_*) = 0$ and the derivative $DF(x_*)$ is invertible. Given any starting point $x_0$, consider the full Newton iteration scheme

$$x_{k+1} = G(x_k) \quad \text{where} \quad G(x) = x - DF(x)^{-1}F(x).$$

Here we prove that if $x_0$ is sufficiently close to $x_*$, then $x_k \to x_*$ as $k \to \infty$.

(**a**) Show that $G(x_*) = x_*$, $DG(x_*) = 0$ and that there is a closed ball $B$ about $x_*$ such that $\|DG(x)\| \leq \frac{1}{2}$ for all $x \in B$. [Hint: You do *not* need to compute $DG(x)$, only $DG(x_*)$.]

(**b**) Show that $G(x) \in B$ for all $x \in B$.

(**c**) Show that $G : B \to B$ is a contraction.

(**d**) Prove that $x_k \to x_*$ as $k \to \infty$ for any $x_0 \in B$.