Preliminary exam, Numerical Analysis, Part 2, differential equations
3:00-4:30 PM, August 18,2020

1. The 2-point boundary value problem,

$$\begin{cases} \dfrac{d^4y}{dx^4} = f(x), \qquad 0 < x < 1 \\ y(0) = y(1) = y''(0) = y''(1) = 0 \end{cases}$$

can be seen as an approximation of a simply supported beam.
(a) Rewrite the problem on variational form and propose suitable spaces for the continuous problem and corresponding discrete FEM approximation. Discuss convergence.
(b) Rewrite the problem as a first order system and determine the corresponding trapezoidal rule FDM approximation.
(c) Describe how this first order system can be solved by initial value techniques (shooting).

2. Consider the heat equation,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot \sigma(x,y)\nabla u, \qquad 0 < \sigma_1 < \sigma(x,y) \le \sigma_2, \qquad 0 < x < 1, 0 < y < 1, \\ u(x,0) = u(x,1) = 0, \quad 0 \le x \le 0 \\ \dfrac{\partial u}{\partial x}(0,y) = \dfrac{\partial u}{\partial x}(1,y) = 0, \quad 0 < y < 1 \\ u(x,y,0) = u_0(x,y), \quad 0 < x < 1, 0 < y < 1 \end{cases}$$

(a) Formulate an implicit Euler-in-time FEM approximation of this problem based on an appropriate variational formulation.
(b) Discuss convergence for the related stationary problem (t-derivative replaced by $f(x,y)$).
(c) Determine the convergence condition (CFL number) for a FDM approximation based on forward Euler in time and centered difference in space. Use von Neumann analysis and assume periodic boundary conditions with constant conductivity σ.

3. The following nonlinear hyperbolic conservation law is given,

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x} f(u) = 0, \qquad 0 \le f'(u) \le C, \qquad t > 0, 0 < x < 1 \\ \textit{periodic boundary conditions} \\ u(x,0) = u_0(x), 0 < x < 1 \end{cases}$$

(a) Formulate an explicit first order finite volume approximation
(b) Show that the scheme is on discrete conservation form and give conditions on the step sizes such that the scheme is monotone.
(c) Formulate a P1 discontinuous Galerkin (DG) approximation with appropriate interface conditions.