ALGEBRA PRELIMINARY EXAM: PART I

In multi-part problems, you may use earlier parts even if you have not done them.

Problem 1

Let R be a PID, π an irreducible element of R and consider the subset M of R^2 of pairs (x, y) with π^2 dividing y and π^3 dividing $y - x\pi^2$.

a) Show that M is a submodule of R^2 of rank 2.
b) Find a basis $\{v_1, v_2\}$ of R^2 and $r_1, r_2 \in R$ with r_1 dividing r_2 such that $r_1 v_1, r_2 v_2$ is a basis of M.

Problem 2

Throughout this problem, G will always be a group of order 27, but not necessarily the same group in each part.

a) Show that if G has a subgroup H of order three which is not normal, G is isomorphic to a subgroup of S_9.
b) Suppose $x \in S_9$ is an element of order 9. Find the orders of $C_{S_9}(\langle x \rangle)$ and $N_{S_9}(\langle x \rangle)$, the centralizer and normalizer of the cyclic subgroup.
c) Suppose $x \in S_9$ is an element of order 9. Describe the 3-Sylow subgoup of $N_{S_9}(\langle x \rangle)$.
d) Up to isomorphism, there are four groups of order 27 which contain an element of order 9. List any that can be embedded in S_9 and justify why your list is correct. [If you desire the classification, two of the groups are abelian and the two non-abelian groups are $\mathbb{Z}/9\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ and $(\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z})/ < (3, 3) >$, the order 27 analog of the quaternions.

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Problem 3

Given a linear transformation $T : V \to V$, we say a subspace W of V is T-stable if $T(W) \subseteq W$. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation which is multiplication by a matrix A which is a Jordan block with eigenvalue $\lambda \neq 0$.

a) Find a proper ascending chain of T-stable subspaces $(0) = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_n = \mathbb{R}^n$.
b) Let W be a T^2-stable subspace such that $W \subseteq W_{k+1}$, $W \nsubseteq W_k$ for some $k > 0$. Show there is a T^2-stable subspace $W' \subset W$ such that $W' \subseteq W_k$, $W' \nsubseteq W_{k-1}$.
c) Show the T^2-stable subspaces are linearly ordered.
d) Show that if the Jordan canonical form of a transformation $T' : \mathbb{R}^n \to \mathbb{R}^n$ is not a Jordan block, the T'-stable subspaces are not linearly ordered.
e) What is the Jordan canonical form of T^2?