ALGEBRA PRELIMINARY EXAM: PART II

Problem 1

Let $\mathbb{F}_3$ be the finite field with 3 elements.

a) Prove that for every positive integer d there exists an irreducible polynomial $f(x) \in \mathbb{F}_3(x)$ of degree d.
b) Determine the number of irreducible polynomials of degree 4 over $\mathbb{F}_3$.

Problem 2

Consider $f(x) = x^4 - 14x^2 + 9 \in \mathbb{Q}[x]$ and let α be a root of $f(x)$.

a) Prove that $f(x)$ is irreducible over $\mathbb{Q}$.
b) Prove that the extension $\mathbb{Q}[\alpha]/\mathbb{Q}$ is Galois.
c) Determine the Galois group of the splitting field of $f(x)$ over $\mathbb{Q}$ as a subgroup of S_4.

Problem 3

a) Let $F/\mathbb{Q}$ be a finite extension. Prove that there exists $\alpha \in F$ such that $F = \mathbb{Q}(\alpha)$ (i.e. $F/\mathbb{Q}$ is a simple extension).
b) Give an example of a finite extension which is not simple (proof required).

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