PRELIMINARY EXAMINATION:
APPLIED MATHEMATICS — Part I
January 16, 2020, 1:00-2:30

Work all 3 of the following 3 problems.

1. Let X be a Banach space with dual space X^* and duality pairing $\langle \cdot, \cdot \rangle$, and let $A, B : X \to X^*$ be linear maps.

(**a**) State the Closed Graph Theorem and what it means for an operator to be closed.

(**b**) Assuming $\langle Ax, y \rangle = \langle Ay, x \rangle$ for all $x, y \in X$, show that A is bounded.

(**c**) Assuming $\langle Bx, x \rangle \geq 0$ for all $x \in X$, show that B is bounded. [Hint: Suppose B is not continuous at 0, so $x_n \to 0$ but $Bx_n \to y \neq 0$. For $w \in X$ such that $\langle y, x \rangle > 0$, consider $x_n + \epsilon w$.]

2. Let $\Omega = [0, 1]$ and $1 \leq p < \infty$ be given and consider the sequence of functions $g_n \in L^p(\Omega)$ defined by $g_n(x) = n^{1/p} e^{-nx}$. Show that:

(**a**) g_n converges pointwise to zero in Ω for any $p \geq 1$;

(**b**) g_n does not converge strongly to zero in $L^p(\Omega)$ for any $p \geq 1$;

(**c**) g_n converges weakly to zero in $L^p(\Omega)$ if $p > 1$, but not if $p = 1$.

3. Prove the Mazur Separation Lemma, which says that if X is a normed linear space, Y a linear subspace of X, $w \in X$ but $w \notin Y$, and

$$d = \operatorname{dist}(w, Y) = \inf_{y \in Y} \|w - y\|_X > 0,$$

then there exists $f \in X^*$ such that $\|f\|_{X^*} \leq 1$, $f(w) = d$, and $f(z) = 0$ for all $z \in Y$. [Hint: Begin by working in $Z = Y + \mathbb{F}w$.]

PRELIMINARY EXAMINATION:
APPLIED MATHEMATICS — Part II

Jan 16, 2020, 2:40–4:10 p.m.

Work all 3 of the following 3 problems.

1. Let $\Omega = (0,1)^2$ and consider the boundary value problem (BVP)

$$\begin{aligned} -u_{xx} + u_{xy} - u_{yy} &= f \quad \text{in } \Omega, \\ -u_x + u_y - u &= g \quad \text{on } \Gamma_L = \{(0,y) : y \in (0,1)\}, \\ u &= 0 \quad \text{on } \Gamma_* = \partial\Omega \setminus \Gamma_L. \end{aligned}$$

Let $H = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_*\}$, which is a Hilbert space.

(**a**) Find the corresponding variational problem for $u \in H$ and test functions $v \in H$. Also give the function spaces containing f and g.

(**b**) Show the general Poincaré type inequality: There exists $\gamma > 0$ such that

$$\|\nabla v\|^2_{L^2(\Omega)} + \int_{\Gamma_L} v^2 \geq \gamma \|v\|^2_{L^2(\Omega)} \quad \forall v \in H.$$

(**c**) Show that there is a unique solution to the variational problem.

2. For fixed $T > 0$, let $g : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be continuous and Lipschitz continuous in the second argument, i.e., there is some $L > 0$ such that

$$\|g(t,v) - g(t,w)\| \leq L\,\|v - w\| \quad \forall v, w \in \mathbb{R}^d, t \in [0,T],$$

where $\|\cdot\|$ is the norm on $\mathbb{R}^d$. For any $u_0 \in \mathbb{R}^d$, consider the initial value problem (IVP) $u'(t) = g(t, u(t))$ and $u(0) = u_0$.

(**a**) Write this IVP as the fixed point of a functional $G : C^0([0,T];\mathbb{R}^d) \to C^0([0,T];\mathbb{R}^d)$.

(**b**) Normally, we use the $L^\infty([0,T])$-norm for $C^0([0,T];\mathbb{R}^d)$. Show that the function $|||\cdot||| : C^0([0,T];\mathbb{R}^d) \to [0,\infty)$, defined by

$$|||v||| = \sup_{0 \leq t \leq T} \big(e^{-Lt}\|v(t)\|\big),$$

is a norm equivalent to the $L^\infty([0,T])$-norm.

(**c**) In terms of this new norm, show that G is a contraction.

(**d**) Explain how we conclude that there is a unique solution $u \in C^1([0,\infty);\mathbb{R}^d)$ to the IVP for all time.

3. Consider finding extremals to the problem: Find $u, v \in C^1_{0,1}([0,1])$ minimizing

$$F(u,v,u',v') = \int_0^1 \big((u')^2 + (v')^2 + 2uv\big)\,dx.$$

(**a**) Find the Euler-Lagrange (EL) equations for this problem.

(**b**) Reduce the EL equations to a single equation and find its solution. [Hint: The fourth roots of unity are ± 1 and $\pm i$.]

(**c**) Find the extremal to the problem, up to solving a 4×4 system of linear equations.

(**d**) If we add the constraint that $\displaystyle\int_0^1 u^2 v'\,dx = 0$, what EL equations do we get?