THE UNIVERSITY OF TEXAS AT AUSTIN
DEPARTMENT OF MATHEMATICS

The Preliminary Examination in Probability
Part I

Wed, Jan 15, 2020

Part I

Problem 1. Let X be a nonnegative random variable. Show that

$$\mathbb{E}[X \log^+(X)] < \infty \iff \int_1^\infty \int_1^\infty \mathbb{P}[X > uv] \, du \, dv < \infty,$$

where $\log^+(x) = \max(\log(x), 0)$. For extra credit, redo the problem, but with the double integral replaced by the double sum $\sum_{m=1}^\infty \sum_{n=1}^\infty \mathbb{P}[X > mn]$.

Problem 2. Let U be a random variable, uniformly distributed on $(0, 1)$. Is it possible to find a sequence of functions f_n such that the sequence $(U, f_n(U))$ converges in distribution to (U, V) where V is also uniform on $(0, 1)$ and U, V are independent? Give an argument or an example why the answer is no, respectively yes.

Problem 3. Assume that X, Y are two random variables, defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $X \in \mathbb{L}^p$ and $Y \in \mathbb{L}^q$, where $p, q \in [1, \infty]$ are conjugate exponents. Prove that for any sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$ we have

$$\mathbb{E}\big[X \mathbb{E}[Y|\mathcal{G}]\big] = \mathbb{E}\big[\mathbb{E}[X|\mathcal{G}] Y\big].$$