PRELIMINARY EXAMINATION IN ALGEBRA
PART I
AUGUST 17, 2020

Please solve at least 3 of the following 4 problems.

(1) Let G be a group of order p^k for some prime p and $k \geq 1$. Show that for every $1 \leq l \leq k$ that G has a normal subgroup of order p^l. Please prove this from first principles.

(2) Let n be an odd number so that $\pi = (1, 2, \ldots, n) \in A_n$. Is the S_n-conjugacy class of π the same as its A_n-conjugacy class?

(3) Let R be a PID. An ideal $I \subset R$ is **primary** if for all $a, b \in R$ with $ab \in I$ either $a \in I$ or there exists $n \in \mathbb{N}$ such that $b^n \in I$. Prove that if $I \subset R$ is primary then there exists a prime element $p \in R$ and $n \in \mathbb{N}$ such that $I = (p^n)$.

(4) Consider $R = M_2(\mathbb{F}_{19})$, the ring of 2×2 matrices over the field with 19 elements. Find a complete set of representatives for the conjugacy classes of order 5 elements. Hint: in $\mathbb{F}_{19}[x]$,

$$x^5 - 1 = (x - 1)(x^2 - 4x + 1)(x^2 + 5x + 1).$$