# ALGEBRA PRELIMINARY EXAM: PART II

## Problem 1

Let $K$ be a field and $f(x) \in K[x]$ be a separable irreducible polynomial of degree 5. Assume that $\alpha, \beta$ are two distinct roots of $f(x)$ such that $K(\alpha) = K(\beta)$. Prove that $K(\alpha)/K$ is Galois.

Hint: Consider the action of the Galois group of $f(x)$ over $K$ on the fields $K(\alpha)$ and $K(\beta)$.

## Problem 2

Let $K/\mathbb{Q}$ be an extension of degree $n$.

i) Show that the number of subfields of $K$ is at most $2^{n!}$.
ii) Suppose that $K = \mathbb{Q}(\alpha, \beta)$. Prove that there exists $m \in \mathbb{Z}$ such that $0 \leq m \leq 2^{n!}$ and $K = \mathbb{Q}(\alpha + m\beta)$.

Hint: Use the Galois correspondence between intermediate fields and subgroups.

## Problem 3

Let $p$ be a prime and $\mathbb{F}_p$ be the field with $p$ elements.

i) Determine the number of irreducible quadratic polynomials in $\mathbb{F}_p[x]$.
ii) Let $f(x)$ be an irreducible quadratic polynomial in $\mathbb{F}_p[x]$ and $K$ be a field of cardinality $p^3$. Prove that $f(x)$ is irreducible in $K[x]$.

*Date*: August 17, 2020.