Real Analysis August 2020 Prelim Exam

Problem 1: Let μ be a finite measure on a σ-algebra $\mathcal{M}$, and let $\{E_t\}_{t>0}$ be a family of elements of $\mathcal{M}$ indexed over $t \in (0,\infty)$. Show that if

$$\mu\Big(\bigcup_{t>0} E_t\Big) < \infty\,,$$

then $\mu(E_t) = 0$ for all but countably many values of t.

Problem 2: Let $(X,\mathcal{M},\mu)$ be a finite measure space and let $f : X\times(-1,1) \to \mathbb{R}$ be a function $f = f(x,t)$ such that for each $t \in (-1,1)$, $f(\cdot,t) : X \to \mathbb{R}$ is $\mathcal{M}$-measurable, and such that for μ-a.e. $x \in X$, $f(x,\cdot)$ has a classical derivative at $t = 0$ in the sense that

$$\frac{\partial f}{\partial t}(x,0) = \lim_{h\to 0^+} \frac{f(x,h)-f(x,0)}{h}$$

exists for μ-a.e. $x \in X$. Show that if there exists $M < \infty$ such that

$$|f(x,t) - f(x,0)| \le M\,|t| \qquad \text{for } \mu\text{-a.e. } x \in X$$

then the function

$$g(t) = \int_X f(x,t)\,d\mu(x)$$

is differentiable at $t = 0$ with

$$g'(0) = \int_X \frac{\partial f}{\partial t}(x,0)\,d\mu(x)\,.$$

Problem 3: Let $\mu_1 = \#$ be the counting measures on $\mathbb{R}$ (so that $\#(E)$ equals the number of elements of E), and let $\mu_2 = \mathcal{L}^1$ be the Lebesgue measure on $\mathbb{R}$. Let $E = \{(x,y) \in \mathbb{R}^2 : 0 \le x = y \le 1\}$. Show that the iterated integrals

$$\int_{\mathbb{R}} d\mu_1(x) \int_{\mathbb{R}} f(x,y)\,d\mu_2(y)\,, \qquad \int_{\mathbb{R}} d\mu_2(x) \int_{\mathbb{R}} f(x,y)\,d\mu_1(y)\,,$$

for $f = 1_E$, the characteristic function of E, are well-defined, but are not equal. Explain why this is not in contradiction to Fubini's theorem.

Problem 4: Given $f : \mathbb{R}^n \to \mathbb{R}$, let $\tau_M(f) = 1_{B_M(0)} \min\{M, \max\{f,-M\}\}$ for $M > 0$. Show that $\tau_M(f) \to f$ in $L^p(\mathbb{R}^n,\mu)$ as $M \to \infty$ whenever $p \in [1,\infty)$,

$f \in L^p(\mathbb{R}^n, \mu)$ and μ is a locally finite Borel measure on $\mathbb{R}^n$. Does this result hold if $p = +\infty$?

Problem 5: Let L be a bounded linear map from a Banach space X to itself, and assume that L is a contraction, that is, let $||L|| < 1$. Define a sequence $\{x_k\}_k$ in X by the recursive relation $x_{k+1} = Lx_k$. Show that $\{x_k\}_k$ is a Cauchy sequence in X (by using the convergence of the geometric series defined by $||L||$) and deduce the existence of a fixed point of L, that is, the existence of $x \in X$ such that $x = Lx$.