PRELIMINARY EXAMINATION IN ANALYSIS

Complex Analysis

August 19, 2020, 1:00-2:30 p.m.

1. Is there an entire function f that satisfies $|f(z)| \geq e^{|z|}$ for all values z large enough ? Either provide an example, or prove that none exists.

2. Assume that f is analytic outside the disk $\{z \in \mathbb{C} : |z| \leq 1\}$ and takes its values inside this disk. Prove that $|f'(2)| \leq 1/3$.

3. Suppose f is analytic on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and satisfies $|f(z)| \leq M$ for all $z \in \mathbb{D}$. Assume further that $f(z)$ vanishes at the points $\{z_j\}_{j=1}^{N}$ where $1 \leq N \leq \infty$.

(**a**) Prove that

$$|f(z)| \leq M \left| \prod_{j=1}^{m} \frac{z - z_j}{1 - \bar{z}_j z} \right| \qquad \forall z \in \mathbb{D},$$

for any $1 \leq m \leq N$ (or if $N = \infty$, then $1 \leq m < N$).

(**b**) If $N = \infty$ and $f \not\equiv 0$, then show that

$$\sum_{j=1}^{\infty} (1 - |z_j|) < \infty.$$

4. Prove that

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$