Applied Mathematics Preliminary Exam, Part A

August 24, 2020

Work 3 of the following 4 problems.

1. Let $s \mapsto T_s$ be a family of bounded linear operators on a Banach space X, indexed by points s in a metric space S. Assume that the function $s \mapsto \|T_s x\|$ is bounded on S, for every $x \in X$. Given $\sigma \in S$, show that the set of all $x \in X$ for which $T_s x \to T_\sigma x$ as $s \to \sigma$, is a closed subspace of X.

2. Let X be an infinite-dimensional Banach space whose dual X' is separable. Prove that there exist $x_1, x_2, x_3, \ldots \in X$ such that $\|x_n\| \to 1$ and $x_n \rightharpoonup 0$ in the weak topology.

3. Let $A : X \to Y$ be a linear operator from a normed vector space X to a Hilbert space Y. Show that A is compact if and only if there exist a sequence of finite rank operators $A_1, A_2, A_3, \ldots$ from X to Y such that $\|A - A_n\| \to 0$ as $n \to \infty$.

4. Let P be a linear operator on a Banach space, satisfying $P^2 = P$. Show that the operator P is continuous if and only if its null space and range are both closed.