PRELIMINARY EXAMINATION:
APPLIED MATHEMATICS — Part II

August 24, 2020

Work all 3 of the following 3 problems.

1. Suppose that $f \in \mathcal{S}$ (the Schwartz space) and α is a multi-index.

(**a**) Prove that $(D^\alpha f)^\wedge(\xi) = (i\xi)^\alpha \hat{f}(\xi)$. [Hint: Use integration by parts.]

(**b**) Prove that $D^\alpha \hat{f}(\xi) = \big((-ix)^\alpha f(x)\big)^\wedge(\xi)$. [Hint: Prove the result for a single derivative and then use iteration.]

2. Suppose that X and Y are Hilbert spaces and $A : X \to X$, $B : Y \to X$, and $C : Y \to Y$ are bounded linear operators, with A being invertible, B being bounded below, and C being negative semi definite. Consider the problem

$$\begin{aligned} Ax + By &= f, \\ B^*x + Cy &= g, \end{aligned}$$

where $f \in X$ and $g \in Y$.

(**a**) Why is A^{-1} continuous and the second equation well posed?

(**b**) Rewrite the problem by using the first equation to solve for x and then replace x in the second equation.

(**c**) Use the Lax-Milgram theorem to show that there is a unique solution to the problem.

3. Let $g : \mathbb{R} \to \mathbb{R}$ be a contraction with fixed point $u_* = g(u_*)$. Consider the problem of finding a continuous function $u : \mathbb{R} \to \mathbb{R}$ near u_* such that

$$u(t) = g(u(t)) + \epsilon \int_0^t (u(s))^3\, ds = G(u),$$

where $\epsilon \geq 0$. Note that $G(u) : C([0,T]) \to C([0,T])$ for any $T > 0$.

(**a**) Use the Banach contraction mapping theorem to show that there is a solution to $u = G(u)$ near u^* for any ϵ, at least for small enough T.

(**b**) Can your solution from (a) be extended so that $T \to \infty$? Why or why not?

(**c**) Suppose that g is C^1 and $Dg(u_*)$ is invertible. Use the implicit function theorem to show that there is a solution near u_* for any fixed T, provided $\epsilon > 0$ is small enough.

(**d**) Can your solution from (c) be extended so that $T \to \infty$? Why or why not?