The University of Texas at Austin
Department of Mathematics

The Preliminary Examination in Probability
Part I

Fri, Aug 21, 2020

Problem 1. Let X be a nonnegative random variable. Show that

$$\mathbb{E}[X\log^+(X)] < \infty \iff \int_1^\infty \int_1^\infty \mathbb{P}[X > uv]\,du\,dv < \infty,$$

where $\log^+(x) = \max(\log(x), 0)$. For extra credit, redo the problem, but with the double integral replaced by the double sum $\sum_{m=1}^\infty \sum_{n=1}^\infty \mathbb{P}[X > mn]$.

Problem 2. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}$ such that $\mu_n \stackrel{w}{\to} \mu$, for some probability measure μ on $\mathbb{R}$ and

$$\sup_{n\in\mathbb{N}} |\varphi_n| \in \mathbb{L}^1(\lambda),$$

where λ is the Lebesgue measure on $\mathbb{R}$ and φ_n is the characteristic function of μ_n. Show that $\mu_n \ll \lambda$ for each $n \in \mathbb{N}$, $\mu \ll \lambda$ and $\frac{\mathrm{d}\mu_n}{\mathrm{d}\lambda} \to \frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$, λ-a.e.

Problem 3. Let X be bounded random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$, and let $\mathbb{Q}$ be a probability measure on $\mathcal{F}$, absolutely continuous with respect to $\mathbb{P}$. Is the following

$$\mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}] = \mathbb{E}[\tfrac{d\mathbb{Q}}{d\mathbb{P}} X|\mathcal{G}] \text{ a.s.} \tag{1}$$

always true? If so, prove it. If not, fix the right-hand side of (1), but without using any (conditional) expectations under $\mathbb{Q}$.