Algebraic Topology Prelim, August 2020

Do all 3 questions. Any theorems used must be clearly stated.

1. A space X has the *fixed point property* (FPP) if every continuous function $f : X \to X$ has a fixed point. Show that

(a) $\mathbb{R}P^2$ has the FPP.

(b) $\mathbb{R}P^3$ does not have the FPP.

2. Let A and B be groups and let G be an index 2 subgroup of the free product $A * B$. Show that G is isomorphic to either

(a) $A * A * B'$, where B' is an index 2 subgroup of B, or

(b) $A' * B * B$, where A' is an index 2 subgroup of A, or

(c) $A' * B' * \mathbb{Z}$, where A' and B' are index 2 subgroups of A and B, respectively.

If you like you may assume that A and B are finitely presented.

3. (a) Let X be a CW-complex with subcomplexes A and B such that $X = A \cup B$. Suppose the map on singular homology $H_q(A \cap B) \to H_q(B)$ induced by inclusion is an isomorphism for all q. Show that inclusion $A \to X$ induces an isomorphism $H_q(A) \cong H_q(X)$ for all q.

(b) Show that the conclusion in (a) is false if the hypotheses are changed by replacing "subcomplexes" by "subspaces". (Hint: Take $X = S^1$.)