Preliminary Examination in Differential Topology
August 2020

Attempt all three questions. Clearly state any theorems you use.

Question 1

Suppose that N_1 and N_2 are two submanifolds of an n-manifold M, of complementary dimension, intersecting transversely at a point $p \in M$ (i.e., $T_pM = T_pN_1 + T_pN_1$). Prove that there exist open neighborhoods $U' \subset \mathbb{R}^n$ and $U \subset M$ of 0 and p respectively, and a diffeomorphism

$$\phi\colon U' \xrightarrow{\cong} U, \quad \phi(0) = p,$$

such that

$$\phi^{-1}(N_1) = V_1 \cap U', \quad \phi^{-1}(N_2) = V_2 \cap U'$$

for vector subspaces V_1, V_2 of $\mathbb{R}^n$.

Question 2

If E is a positive-definite real inner product space, of dimension n, define $V_k(E)$ to be the set of k-tuples $(v_1, \ldots, v_k) \in E^k$ of mutually orthogonal unit vectors in E.

(a) Explain how to give $V_k(E)$ the structure of smooth manifold, and (when it is non-empty) calculate its dimension. If $i\colon E' \to E$ is the inclusion of a vector subspace, prove that the induced map

$$V_k(E') \to V_k(E), \quad (v_1, \ldots, v_k) \mapsto (i(v_1), \ldots, i(v_k)),$$

is a smooth embedding.

(b) If E_1 and E_2 are vector subspaces of $\mathbb{R}^n$ intersecting transversely, show that the images of $V_k(E_1)$ and $V_k(E_2)$ in $V_k(\mathbb{R}^n)$ are transversely intersecting submanifolds.

Question 3

On $\mathbb{C}^2$ with complex coordinates $(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)$, define the 2-form $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Prove the *non-existence* of each of the following:

(a) a (C^∞) diffeomorphism $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ such that $F^*\omega = \omega$, mapping some open ball of radius 2 into an open ball of radius 1 [*hint:* consider $\omega \wedge \omega$];

(b) a C^∞ map $g\colon \Sigma \to \mathbb{C}^2$, where Σ is a compact surface without boundary, such that $g^*\omega$ is nowhere-vanishing [*hint:* Stokes's theorem];

(c) a diffeomorphism $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ such that $F^*\omega = \omega$ and $F(L) = M$, where

$$L = \{z \in \mathbb{C}^2 : x_1^2 + y_1^2 = 1,\ x_2^2 + y_2^2 = 1\},$$
$$M = \{z \in \mathbb{C}^2 : x_1^2 + x_2^2 = 1,\ y_1^2 + y_2^2 = 1\}.$$