Preliminary Examination in Algebra, Part II
January 2019

(1) Let α be the real, positive fourth root of 5. Set $F = \mathbb{Q}(\alpha)$ and let E be the normal closure of F.

(a) Determine the Galois group $\mathsf{Gal}(E/\mathbb{Q})$ as an abstract group.

(b) Prove that F is not a subfield of any cyclotomic extension of $\mathbb{Q}$.

(c) Describe, in terms of α and $i := \sqrt{-1}$, all subfields of E which are normal over $\mathbb{Q}$.

(2) Let k be a finite field.

(a) Describe, without proof, the structure of the multiplicative group $k^{\times}$.

(b) Prove that every element of k is a sum of two squares.

(3) Let p be a prime, and $\mathbb{F}_p$ the field with p elements. Let x and y be algebraically independent indeterminates over $\mathbb{F}_p$. Consider the field $L = \mathbb{F}_p(x, y)$ and its subfield $K = \mathbb{F}_p(x^p, y^p)$.

(a) Determine the separable and inseparable degrees of L/K.

(b) Prove that L/K is not a simple extension.