PRELIMINARY EXAMINATION IN ANALYSIS
Part II, Complex Analysis
January 15, 2019

Work 4 *of the following* 5 *problems.*

1. Given $z \in \mathbb{C}$ and a smooth closed curve γ in $\mathbb{C} \setminus \{z\}$, denote by $n(\gamma, z)$ the index or winding number of γ about z. If γ admits a representation

$$\gamma(t) = \sum_{k=-N}^{N} c_k e^{ikt}\,, \qquad t \in [0, 2\pi]\,,$$

with c_{-N} and c_N not both zero, show that $-N \leq n(\gamma, z) \leq N$.

2. Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\sin \pi x}{x^2 - x}\, dx$. Explain all steps carefully.

3. Assume that f is meromorphic on $\mathbb{C}$ and bounded outside some compact set. Show that f is a rational function.

4. Prove that

$$\frac{d}{dz}\left(\frac{\pi \sin(z)}{\sin(\pi z)}\right) = \sum_{n \in \mathbb{Z}} \frac{(-1)^{n+1} \sin(n)}{(z-n)^2}$$

for all $z \in \mathbb{C} \setminus \mathbb{Z}$, with the right-hand side converging uniformly on every compact subset of $\mathbb{C} \setminus \mathbb{Z}$.

5. Determine all analytic functions $f : H \to \mathbb{C}$ on the half-plane $H = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ that satisfy $f(\sqrt{n}\,) = n$ and $\left|f^{(n)}(1)\right| \leq 3$ for all positive integers n.