PRELIMINARY EXAMINATION: APPLIED MATHEMATICS — Part II

January 18, 2019, 2:40–4:10 p.m.

Work all 3 of the following 3 problems.

1. Let $\mathcal{S}'(\mathbb{R})$ be the space of tempered distributions on $\mathbb{R}$. Under what conditions on the complex sequence $\{a_k\}_{k=1}^{\infty}$ is $\displaystyle\sum_{k=1}^{\infty} a_k\,\delta_k \in \mathcal{S}'(\mathbb{R})$? Here, δ_k is the point mass centered at $x = k$.

2. Show the following two statements about Sobolev spaces, where $\Omega \subset \mathbb{R}^d$ is a domain.

(**a**) There is no embedding of $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \le p \le d$ and $q > dp/(d-p)$. [Hint: Show a counterexample with the function $f(x) = |x|^\alpha$ by choosing an appropriate domain Ω and exponent α.]

(**b**) There is no embedding of $W^{1,p}(\Omega) \hookrightarrow C^0_B(\Omega)$ for $1 \le p < d$. Note that in the previous case, f is not bounded. What can you say about which (negative) Sobolev spaces the Dirac mass lies in?

3. Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary and $\{u_k\} \subset H^{2+\varepsilon}(\Omega)$ is a bounded sequence, where $\varepsilon > 0$.

(**a**) Show that there is $u \in H^2(\Omega)$ such that, for a subsequence, $\{u_{k_j}\}_{k=1}^{\infty} \to u$ in $H^2(\Omega)$.

(**b**) Find all q and $s \ge 0$ such that, for a subsequence, $\{u_{k_j}\} \to u$ in $W^{s,q}(\Omega)$.