THE UNIVERSITY OF TEXAS AT AUSTIN
DEPARTMENT OF MATHEMATICS

The Preliminary Examination in Probability
Part I

Thursday, Jan 17, 2019

Part I

Problem 1. Consider a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ in $\mathbb{L}^1$ and a C^2 function $\varphi:\mathbb{R}\to\mathbb{R}$ with the property that $\varphi''(x)\geq\varepsilon>0$ for all x. Suppose that

$$\mathbb{E}[X_n]\to\mu\in\mathbb{R} \text{ and } \mathbb{E}[\varphi(X_n)]\to\varphi(\mu).$$

Show that $X_n\to\mu$ in probability.

Problem 2. Fix a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and on it a random variable N with the unit Poisson distribution:

$$\mathbb{P}[N=n]=(n!e)^{-1} \text{ for } n\in\mathbb{N}\cup\{0\}.$$

(1) Let $\{Z_n\}_{n\in\mathbb{N}\cup\{0\}}$ be a sequence of random variables independent of N, with characteristic functions $\varphi_n=\varphi_{Z_n}$. Express the characteristic function of the random variable Z_N, given by $Z_N(\omega)=Z_{N(\omega)}(\omega)$ for $\omega\in\Omega$, in terms of the functions $\{\varphi_n\}_{n\in\mathbb{N}}$.

(2) Let $\{X_n\}_{n\in\mathbb{N}}$ be an iid sequence independent of N, with the (common) characteristic function φ. Show how to construct a random variable Y whose characteristic function is $e^{\varphi-1}$, from $\{X_n\}_{n\in\mathbb{N}}$ and N.

Problem 3. Let Z_1 and Z_2 be independent standard normals. Find the conditional density of $e^{Z_1-Z_2}$, given $\sigma(e^{Z_1+2Z_2})$?