THE UNIVERSITY OF TEXAS AT AUSTIN
DEPARTMENT OF MATHEMATICS

The Preliminary Examination in Probability
Part II

Thursday, January 17th, 2019

Problem 1. Let W be a one-dimensional standard Brownian motion. Let μ, σ be constant real numbers and x be an initial value. Solve in closed form the equation

$$\begin{cases} dX_t = \mu X_t dt + \sigma X_t dW_t, \\ X_0 = x. \end{cases}$$

Problem 2. Let W be a standard one-dimensional Brownian motion and M be its' running maximum process, i.e.

$$M_t = \max_{0 \leq s \leq t} W_s, \quad 0 \leq t < \infty.$$

Consider a two-times continuously differentiable function f

$$f : \{(x, m) : m \geq 0, -\infty < x \leq m\} \to \mathbb{R}.$$

Find a necessary and sufficient condition so that the process Y defined by

$$Y_t = f(W_t, M_t), 0 \leq t < \infty,$$

is a local martingale.

Problem 3. Consider a finite time horizon T and a RCLL sub-martingale $(M_t)_{0 \leq t \leq T}$ on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$. Consider the optimization problem of finding the stopping time τ that maximizes the expected value of M at the (random) time τ, namely the problem

$$\sup_{\tau \text{ stopping time}} \mathbb{E}[M_\tau].$$

Find the optimizer τ^*.