ALGEBRA PRELIMINARY EXAM: PART I

In multi-part problems, you may use earlier parts even if you have not done them.

Problem 1

Let k be a field, n a positive integer, and T the linear transformation on k^n defined by

$$T(x_1, x_2, \ldots, x_n) = (x_n, x_1, x_2, \ldots, x_{n-1}).$$

We view k^n as a $k[x]$-module with x acting as T.

a) Show that the $k[x]$-module k^n is isomorphic to $k[x]/(x^n - 1)$.

b) Let V be a linear subspace of k^n satisfying $T(V) \subseteq V$. Prove that there exists a monic polynomial $g(x) \in k[x]$ such that V corresponds to

$$\{g(x)a(x) \mid a(x) \in k[x], \deg a(x) < n - \deg g(x)\}$$

under the above isomorphism.

c) Take $k = \mathbb{R}$, the real numbers, and $n = 3$. Describe explicitly all subspaces V of $\mathbb{R}^3$ satisfying $T(V) \subseteq V$.

Problem 2

Let R be an integral domain and I be an ideal of R. Fix $x \in R$ and define

$$(I : x) = \{r \in R \mid rx \in I\}.$$

a) Prove $(I : x)$ is an ideal of R.

b) Show that if $(I : x) = I$, then $(I : x^2) = I$.

c) Show that if $(I : x) \subseteq xR$, but $(I : x) \nsubseteq \cap_{n=1}^{\infty} x^n R$, then $I \neq (I : x)$.

d) If R is not a principal ideal domain, show that R has an ideal maximal with respect to the property of not being principal.

e) If R is a unique factorization domain with the property that every maximal ideal is principal, show that R is a principal ideal domain.

Date: August 19, 2019.

Problem 3

Assume G is a group of order $456 = 2 \cdot 3 \cdot 7 \cdot 13$.

a) Show G has a normal Sylow subgroup of order either 7 or 13.

b) Show that G is a semidirect product of a cyclic group of order 91 by a group of order 6.

c) Let K be a group of order 42. Then K is the semidirect product of a cyclic group of order 7 by a group of order 6. (Prove this **only** if you are unable to do part (b).) How many non-isomorphic groups of order 42 are there? Roughly describe them.

d) **More challenging** – do not spend too much time.
   (i) Let H be the direct product of two characteristic subgroups H_1 and H_2. Prove $\operatorname{Aut} H \cong \operatorname{Aut} H_1 \times \operatorname{Aut} H_2$.
   (ii) Noting the similarity between $42 = 7 \cdot 6$ and $78 = 13 \cdot 6$, how many non-isomorphic groups of order 456 are there?