ALGEBRA PRELIMINARY EXAM: PART II

Problem 1

Let $\alpha = \sqrt[3]{4} + \sqrt[3]{2} + 1$.

a) Determine the degree of α over $\mathbb{Q}$.

b) Prove that $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha^2)$.

Problem 2

Let p be a prime and $\mathbb{F}_p$ the field with p elements.

a) Describe finite extensions of $\mathbb{F}_p$ (no proofs required).

b) Determine the splitting field over $\mathbb{F}_p$ of $x^p - x - a \in \mathbb{F}_p[x]$ where $a \in \mathbb{F}_p \setminus \{0\}$.

Problem 3

Consider the polynomial $f(x) = x^4 - 2x^2 + 2$. Let L be the splitting field of $f(x)$ over $\mathbb{Q}$.

a) Prove that $f(x)$ is irreducible in $\mathbb{Q}[x]$.

b) Determine the degree of $L/\mathbb{Q}$.

c) Determine the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ as an abstract group.

d) Prove that L is not a subfield of a cyclotomic extension of $\mathbb{Q}$, i.e. $L \not\subseteq \mathbb{Q}(\zeta)$ where ζ is a root of unity.

e) Determine all the subfields of $L/\mathbb{Q}$ which are Galois extensions of $\mathbb{Q}$.

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