PRELIMINARY EXAMINATION:
APPLIED MATHEMATICS — Part I
August 26, 2019, 1:00-2:30

Work all 3 of the following 3 problems.

1. Let Ω be a compact set in $\mathbb{R}^d$ and let $K : \Omega\times\Omega \to \mathbb{R}$ be continuous and symmetric (i.e., $K(x, y) = K(y, x)$). Suppose that $K \geq 0$, and let the operator T be defined by $Tf(x) = \int_\Omega K(x, y)\, f(y)\, dy$.

(**a**) State the spectral theorem for a compact, self-adjoint operator.

(**b**) Show Mercer's Theorem: there is an ON base for $L^2(\Omega)$ consisting of eigenfunctions $\{e_j\}_{j=1}^\infty$ of T with corresponding eigenvalues $\{\lambda_j\}_{j=1}^\infty$ such that each $\lambda_j \geq 0$ and

$$K(x, y) = \sum_{j=1}^\infty \lambda_j\, e_j(x)\, e_j(y).$$

[The sum is absolutely and uniformly convergent in $L^2(\Omega \times \Omega)$, but you need *not* show this fact.]

(**c**) Define $\mathrm{Trace}(T) = \int_\Omega K(x, x)\, dx$ and show that

$$\mathrm{Trace}(T) = \sum_{j=1}^\infty \lambda_j.$$

2. Let H be a Hilbert space with inner product $\langle\cdot, \cdot\rangle$.

(**a**) Prove the parallelogram law: For all $x, y \in H$,

$$\|x + y\|^2 + \|x - y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big).$$

(**b**) Prove the Best Approximation Theorem. That is, if $M \subset H$ is nonempty, convex, and closed, and if $x \in H$, then there is a unique $y \in M$ such that

$$\mathrm{dist}(x, M) = \inf_{z\in M} \|x - z\| = \|x - y\|.$$

3. Let X and Y be NLS's.

(**a**) Show that if a linear operator $S : X^* \to Y^*$ is weakly-$*$ sequentially continuous, that is,

$$f_n \overset{\text{weak-}*}{\longrightarrow} f \text{ in } X^* \Longrightarrow S(f_n) \overset{\text{weak-}*}{\longrightarrow} S(f) \text{ in } Y^*,$$

then S is bounded.

(**b**) Given a linear operator $T : X \to Y$, assume that the dual (or conjugate or adjoint) $T^* : Y^* \to X^*$ is defined. Show that T^* is weakly-$*$ sequentially continuous.

(**c**) Show that whenever $T^* : Y^* \to X^*$ is defined, T^* is bounded.

PRELIMINARY EXAMINATION:
APPLIED MATHEMATICS — Part II

August 26, 2019, 2:40–4:10 p.m.

Work all 3 of the following 3 problems.

1. Let $\Omega \subset \mathbb{R}^2$ be an open, connected, and bounded domain containing 0. Let

$$X = \{f \in W^{1,3}(\Omega) \, : \, f(0) = 0\}.$$

(**a**) Use the Sobolev Embedding Theorem to conclude that X is a Banach space, and $X \neq W^{1,3}(\Omega)$.

(**b**) Prove the Poincaré-like inequality $\|f\|_{L^3(\Omega)} \leq C\|\nabla f\|_{L^3(\Omega)}$, for some constant C independent of $f \in X$.

2. Suppose that $\Omega \subseteq \mathbb{R}^d$ is a bounded domain with Lipschitz boundary and $\{u_k\}_{k=1}^{\infty} \subset H^{2+\varepsilon}(\Omega)$ is a bounded sequence, where $\varepsilon > 0$.

(**a**) State the Rellich-Kondrachov Theorem. [For the rest of the problem, assume that it holds with nonintegral values for the number of derivatives.]

(**b**) Show that there is $u \in H^{2+\varepsilon}(\Omega)$ such that, for a subsequence, $u_{k_j} \to u$ in $H^2(\Omega)$.

(**c**) Find all q and $s \geq 0$ such that, for a subsequence, $u_{k_j} \to u$ in $W^{s,q}(\Omega)$.

3. Let Ω be a domain with a smooth boundary. Consider the differential problem

$$\begin{aligned} p - \nabla \cdot a\nabla p - \nabla \cdot b\nabla q + d(p - q) &= 0 \quad \text{in } \Omega, \\ -\nabla \cdot c\nabla q + d(q - p) &= f \quad \text{in } \Omega, \\ -(a\nabla p + b\nabla q) \cdot \nu &= g \quad \text{on } \partial\Omega, \\ q &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where a, b, c, and $d \geq 0$ are bounded, smooth functions, $f \in H^{-1}(\Omega)$, and $g \in H^{-1/2}(\partial\Omega)$. Moreover, assume that there is some $\gamma > 0$ such that $a \geq \gamma$, $c \geq \gamma$, and $|b| \leq \gamma$.

(**a**) Define a suitable variational problem for the differential equations. Be sure to identify your function spaces for p, q, and the test functions.

(**b**) Show that there is a unique solution to the variational problem.