Preliminary exam in Differential Topology

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Attempt all three questions.

Question 1

(a) Prove that any compact manifold M embeds into $\mathbb{R}^N$, for some N.

(b) For $v \in S^{N-1}$, let $p_v\colon \mathbb{R}^N \to v^\perp \subset \mathbb{R}^N$ be orthogonal projection away from v, i.e., $p_v(x) = x - (x \cdot v)v$. Suppose that $M \subset \mathbb{R}^N$ is an n-dimensional submanifold. Using the standard norm on $\mathbb{R}^N$, define

$$S(TM) = \{(x, u) \in M \times T_x M : \|u\| = 1\},$$
$$\pi\colon S(TM) \to S^{N-1}, \quad (x, u) \mapsto u,$$
$$\sigma\colon \{(x, y) \in M \times M : x \neq y\} \to S^{N-1}, \quad \sigma(x, y) = \frac{x - y}{\|x - y\|}.$$

Prove that $p_v|_M\colon M \to v^\perp$ is an injective immersion if and only if v does not lie in $\operatorname{im}\sigma \cup \operatorname{im}\pi$.

(c) Prove that any compact n-manifold M embeds into $\mathbb{R}^{2n+1}$.

Question 2

Let $\beta \in \Omega_c^k(\mathbb{R}^n)$ be a differential k-form of compact support on $\mathbb{R}^n$, where $n > 0$ and $k > 0$. Prove that the following are equivalent:

(i) There exists a compactly supported $(k-1)$-form $\alpha \in \Omega_c^{k-1}(\mathbb{R}^n)$ with $d\alpha = \beta$.

(ii) Either $k \neq n$, or $k = n$ and $\int_{\mathbb{R}^n} \beta = 0$.

[*Hint:* Extend β to S^n.]

Question 3

For n a positive integer, let $G = GL_n(\mathbb{C})$ be the group of invertible $n \times n$ complex matrices, regarded as a manifold. Let $H = \{g \in G : \det g \in \mathbb{R}\}$. Let $SL_n(\mathbb{C})$ be the special linear group and $U(n)$ the unitary group.

(i) Prove that H, $SL_n(\mathbb{C})$ and $U(n)$ are submanifolds of G.

(ii) Determine which of the following intersections of submanifolds are transverse:

  (a) $U(n) \cap SL_n(\mathbb{C})$.

  (b) $U(n) \cap H$.