The University of Texas at Austin
Department of Mathematics

# The Preliminary Examination in Probability
# Part I

**Thursday, Aug 23, 2018**

## Part I

**Problem 1.** Let $(S, \mathcal{S})$, $(T, \mathcal{T})$ and $(R, \mathcal{R})$ be (nonempty) measurable spaces. Consider the following two statements for the function $f : S \times T \to R$:

(1) $x \mapsto f(x, y)$ is measurable for each $y \in T$ and $y \mapsto f(x, y)$ is measurable for each $x \in S$.
(2) $f$ is a measurable function

Which of the two implications (1) $\to$ (2) and (2) $\to$ (1) is/are true in general? (*Note:* Give a rigorous proof if a statement is true, or a counterexample, together with the proof that it is, indeed, a counterexample, if a statement is false.)

**Problem 2.** Given $N \in \mathbb{N}$, let $X_1, \dots, X_N$ be independent, $\mathbb{N} \cup \{0\}$-valued random variables such that the sum $Y = X_1 + \dots + X_N$ is binomially distributed with parameters $n \in \mathbb{N} \cup \{0\}$ and $p \in (0, 1)$, i.e., such that $\mathbb{P}[Y = k] = \binom{n}{k} p^k (1-p)^{n-k}$ for $k = 0, \dots, n$. Show that there exist constants $n_1, \dots, n_N \in \mathbb{N} \cup \{0\}$ such that $n_1 + \dots + n_N = n$ and $X_i$ is binomially distributed with parameters $n_i$ and $p$, for each $i = 1, \dots, N$.

**Problem 3.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{G}$ a sub-$\sigma$-algebra of $\mathcal{F}$ and $\{A_n\}_{n \in \mathbb{N}}$ a sequence $\mathcal{G}$-conditionally independent random variables. Show that

$$\Big\{ \sum_i \mathbb{P}[A_i | \mathcal{G}] = \infty \Big\} = \Big\{ \mathbb{P}[\limsup_i A_i | \mathcal{G}] = 1 \Big\}, \text{ a.s.,}$$

where, as usual, two events are equal a.s., if their indicators are a.s.-equal random variables.

(*Hint:* Use the equivalent definition of conditional independence: $\{A_n\}_{n \in \mathbb{N}}$ is an independent sequence under the probability measure $\mathbb{P}_B := \mathbb{P}[\cdot \cap B] / \mathbb{P}[B]$ for each $B \in \mathcal{G}$ with $\mathbb{P}[B] > 0$. You do not have to prove that this definition is equivalent to the classical one.)