# PRELIMINARY EXAMINATION: APPLIED MATHEMATICS — Part I

January 13, 2017, 1:00-2:30

*Work all 3 of the following 3 problems.*

**1.** Let $H$ be a Hilbert space and $P_j : H \to M_j$ be an orthogonal projection onto $M_j$, $j = 1, 2$. Let $N_j = N(P_j)$ be the nullspace of $P_j$.

(**a**) Show that $\|P_j\| \le 1$ and $P_j \ge 0$.

(**b**) Show that the following are equivalent.

i. $P_2 P_1 = P_1 P_2 = P_1$
ii. $\|P_1 x\| \le \|P_2 x\|$ for all $x \in H$
iii. $P_1 \le P_2$
iv. $N_1 \supset N_2$
v. $M_1 \subset M_2$

[Hint: Use the order i $\implies$ ii $\implies$ iii $\implies$ iv $\implies$ v $\implies$ i.]

**2.** Let $X$ and $Y$ be Banach spaces. Let $A : X \to X^*$, $B : Y \to X^*$, and $C : Y \to Y^*$ be bounded linear operators. Suppose that $A$ maps onto $X^*$ and $C$ maps onto $Y^*$, and that there are constants $\alpha > 0$ and $\gamma > 0$ such that

$$Ax(x) \ge \alpha \|x\|_X^2 \quad \text{and} \quad Cy(y) \ge \gamma \|y\|_Y^2 \qquad \forall x \in X,\ y \in Y.$$

Given $f \in X^*$ and $g \in Y^*$, consider the problem

$$\begin{aligned} Ax - By &= f, \\ B^* x + Cy &= g. \end{aligned}$$

(**a**) The notation $B^* x$ is not quite correct. Explain its obvious meaning.

(**b**) Show that $A$ has an inverse and that $\|A^{-1}\| \le 1/\alpha$.

(**c**) Prove that if there exists a solution $(x, y) \in X \times Y$ to the problem, then it is unique. [Hint: Show that $Ax(x) + Cy(y) = f(x) + g(y)$.]

(**d**) If $\|B\| < \sqrt{\alpha\gamma}$, show that there is a solution to the problem.

**3.** Let $I = [0, 1]$ and $A : L^2(I) \to L^2(I)$ be defined by

$$Af(x) = \int_0^1 f(y) \sin\left(\frac{x + y}{2}\right) dy.$$

(**a**) Show that $A$ is compact and self-adjoint.

(**b**) Show that $\|A\| < 1$.

(**c**) Show that the smallest eigenvalue of $A$ is strictly negative. [Hint: Rayleigh Quotient.]