THE UNIVERSITY OF TEXAS AT AUSTIN
DEPARTMENT OF MATHEMATICS

The Preliminary Examination in Probability
Part I

Thursday, January 12th, 2017

Problem 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathbb{L}^0$ be the collection of all $\mathbb{P}$-a.s.-equivalence classes of $\mathbb{R}$-valued random variables. Show that there exists a metric $d : \mathbb{L}^0 \times \mathbb{L}^0 \to [0, \infty)$ such that $(\mathbb{L}^0, d)$ is a complete metric space, and a sequence $\{X_n\}_{n\in\mathbb{N}}$ in $\mathbb{L}^0$ converges under d if and only if it converges in probability.

Problem 2. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}$ such that $\mu_n \stackrel{w}{\to} \mu$, for some probability measure μ on $\mathbb{R}$ and

$$\sup_{n\in\mathbb{N}} |\varphi_{\mu_n}| \in \mathbb{L}^1(\lambda),$$

where λ is the Lebesgue measure on $\mathbb{R}$ and φ_{μ_n} is the characteristic function of μ_n. Show that $\mu \ll \lambda$ and $\mu_n \ll \lambda$ for each $n \in \mathbb{N}$, and $\frac{d\mu_n}{d\lambda} \to \frac{d\mu}{d\lambda}$, λ-a.e.

Problem 3. Let $\xi_1, \xi_2, ...$ be a sequence of independent random variables such that

$$\mathbb{P}\left(\xi_n = \frac{1}{2^n}\right) = \mathbb{P}\left(\xi_n = -\frac{1}{2^n}\right) = \frac{1}{2}, \quad n = 1, 2, \ldots$$

Denote by

$$X_n = \xi_1 + \ldots \xi_n, \quad n = 1, 2, \ldots$$

Show that there exist a random variable X_∞ such that X_n converges a.s. and in L^2 to X_∞. Compute the law of the random variable X_∞. Is the convergence also an L^∞ convergence?