Preliminary exam: Numerical Analysis, Part B, January 13, 2016

Name:____ EID:__________________

1. Consider the ordinary differential equation initial value problem,

$$x'(t) = f(x(t)), \quad t > 0,$$
$$x(0) = x_0$$

and the corresponding two stage Runge-Kutta approximation

$$x_{n+1} = x_n + \alpha_1 h k_1 + \alpha_2 h k_2,$$
$$k_1 = f(x_n), \; k_2 = f(x_n + \beta h k_1),$$
$$x_n \approx x(t_n), \; t_n = nh$$

(a) For which α_1, α_2 and β will the method converge as $h \to 0$?
(b) For which α_1, α_2 and β is the method of second order?
(c) Can a method on this form be A-stable?

Motivate your answers.

2. The following elliptic PDE is given,

$$-\nabla \cdot a(x,y)\nabla u + b \cdot \nabla u + cu = f(x,y), \quad 0 < x < 1, \; 0 < y < 1, \; 0 < a \le a(x,y) \le A$$
$$u = d_1(x,y), \; x = 0 \; and \; x = 1, 0 < y < 1,$$
$$u_y = d_2(x,y), \; y = 0 \; and \; y = 1, 0 < x < 1,$$

(a) Rewrite the equation on weak form.
(b) Show that the relevant bilinear form is continuous and coersive and that the relevant linear form is continuous when $d_1 = d_2 = 0$ for appropriate values of the vector b and constant $c > 0$. Give the fundamental error estimate for a finite element approximation based on the weak form in terms of the best approximation in the space of basis function.
(c) Modify the boundary conditions to be appropriate for $a(x.y) = 0$ and describe a discontinuous Galerkin formulation for this case.

3. A hyperbolic system of nonlinear scalar conservation law has the form,

$$u(x,t)_t + f_1(u(x,t))_x + g_1(u(x,t),v(x,t)) = 0$$
$$v(x,t)_t + f_2(v(x,t))_x + g_2(u(x,t),v(x,t)) = 0$$

(a) Recommend suitable initial and boundary conditions for the hyperbolic system, $(t > 0, a < x < b)$.
(b) Devise an upwind finite difference method for the equation above when $f_1(u) > 0, f_2(v) < 0$ and show that the method is consistent and for, $g_1 = g_2 = 0,$ on conservation form.
(c) Use von Neumann analysis when, $f_1(u) = a_1 u \ (a_1 > 0), f_2(u) = a_2 u \ (a_1 < 0)$, to determine necessary and sufficient conditions for the spatial and temporal step sizes, $\Delta x, \Delta t$, to guarantee L_2 stability.