PRELIMINARY EXAMINATION: APPLIED MATHEMATICS — Part I

August 19, 2016, 1:00-2:30

Work all 3 of the following 3 problems.

1. Prove the Mazur Separation Theorem: Let X be an NLS, Y a linear subspace of X, and $w \in X$, $w \notin Y$. If $d = \operatorname{dist}(w, Y) = \inf_{y \in Y} \|w - y\|_X > 0$, then there exists $f \in X^*$ such that $\|f\|_{X^*} \leq 1$, $f(w) = d$, and $f(y) = 0$ for all $y \in Y$.

2. Let X be a vector space and let W be a vector space of linear functionals on X. Suppose that W separates points of X, meaning that for any $x, y \in X$, $x \neq y$, there exists $w \in W$ such that $w(x) \neq w(y)$. Let X be endowed with the smallest topology such that each $w \in W$ is continuous (we call this the W-weak topology of X).

(**a**) Describe a W-weak open set of 0.

(**b**) Prove that if L is a W-weakly continuous linear functional on X, then $L \in W$. [Hint: Consider the inverse image of $B_1(0) \subset \mathbb{F}$, which must contain a W-weak open set of 0, and apply the result from linear algebra that if w_i, $i = 1, 2, ..., n$, and L are linear functionals on X such that $L(x) = 0$ whenever $w_i(x) = 0$ for all i, then L is a linear combination of the w_i.]

(**c**) Based on this result, if X is an NLS, characterize the set of weak-* continuous linear functionals on X^*.

3. Let $\Omega = (-1, 1)^2 \subset \mathbb{R}^2$ and $T : \mathcal{D}(\Omega) \to \mathcal{D}(-1, 1)$ be defined by $T\varphi(x, y) = \varphi(x, 0)$.

(**a**) Show that T is a (sequentially) continuous linear operator.

(**b**) Note that $T' : \mathcal{D}'(-1, 1) \to \mathcal{D}'(\Omega)$. Determine $T'(\delta_0)$ and $T'(\delta_0')$, where δ_0 is the usual Dirac point distribution in one space dimension at 0.