PRELIMINARY EXAMINATION IN ANALYSIS
PART I - REAL ANALYSIS
JANUARY 9, 2015, 1:00 PM - 2:30 PM

Please solve all of the following four problems.

(1) Let Z be a subset of $\mathbb{R}$ with measure zero. Show that the set $A = \{x^2 \mid x \in Z\}$ also has measure zero.

(2) Let $E \subset \mathbb{R}$ be a measurable set such that $0 < |E| < \infty$. Prove that for every $\alpha \in (0,1)$ there is an open interval I such that

$$|\, E \cap I \,| \geq \alpha |\, I \,|.$$

(3) For any natural number n construct a function $f \in L^1(\mathbb{R}^n)$ such that for any ball $B \subset \mathbb{R}^n$, f is not essentially bounded on B.

(4) Let $g \in L^1(\mathbb{R}^n)$, $\|g\|_{L^1(\mathbb{R}^n)} < 1$. Prove that there is a unique $f \in L^1(\mathbb{R}^n)$ such that

$$f(x) + (f * g)(x) = e^{-|x|^2}, \quad x \in \mathbb{R}^n \quad \text{a.e.}$$