Numerical Analysis Preliminary Exam

January 5, 2015

Part I

1. Consider $A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}$ and its Singular Value Decomposition (SVD).

   (a) Find an SVD of A involving unitary matrices having only real entries and having minimal number of minus signs.

   (b) Find an Eigenvalue decomposition of the matrix

   $$B = \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}.$$

2. Consider the linear least squares problem

   $$\min_x \|Ax - b\|_2, \tag{1}$$

   where $A \in R^{m\times n}$ with $m \geq n$.

   (a) Derive the normal equations for solving (1).

   (b) Show how to use QR decomposition and SVD (singular value decomposition) to solve (1).

   (c) Suppose A does not have full column rank. Is the least squares solution unique? Characterize all solutions in terms of the SVD of A.

   (d) Suppose A does have full column rank, but many of its singular values are small (for example, $m = 100, n = 50, \sigma_1 = 2, \sigma_1, \cdots, \sigma_{25} > 1$ and $\sigma_{26}, \ldots, \sigma_{50} < 10^{-13}$). How will you solve the least squares problem (1) in this case? Discuss.

3. Derive a numerical method with optimal computational efficiency for finding the minimum of $G :\mapsto \mathbb{R}^m \mapsto \mathbb{R}$ which is strictly convex and twice continuously differentiable. If G is quadratic and Newton's method is used, how many steps would be needed to convergence? Justify your answers.