PRELIMINARY EXAMINATION IN ALGEBRAIC TOPOLOGY
JANUARY 2015

Instructions:

- Attempt all three questions.
- Explain your arguments as clearly as possible.
- The exam is 90 minutes in duration.

(1) Show that there does not exist a compact surface F (with boundary) with a connected 2-sheeted covering space $\tilde{F}$ such that $\tilde{F}$ has three boundary components and $H_1(\tilde{F}) \cong \mathbb{Z}^5$.

[Hint: You may quote the fact that if $p\colon \tilde{X} \to X$ is a 2-sheeted covering map, with $\tilde{X}$ path-connected, $x_0 \in X$, and $\tilde{x}_0 \in p^{-1}(x_0)$, then $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is the kernel of some non-trivial homomorphism $\pi_1(X, x_0) \to \mathbb{Z}/2$.]

(2) Recall that the *mapping cylinder* M_f of a map $f : X \to Y$ is the quotient space obtained from the disjoint union of $X \times I$ and Y by identifying $(x, 1)$ with $f(x)$ for all $x \in X$.

Recall also that the *double* of a space X along a subspace A is the quotient space obtained from the disjoint union of two copies of X by identifying the two copies of A via the identity map.

Let $f : S^1 \to S^1 \times S^1$ be the map $f(z) = (z^2, 1)$. (Here we regard S^1 as the set of complex numbers of absolute value 1.) Let V be the double of M_f along $S^1 \times \{0\} \subset S^1 \times I$. Compute the homology groups $H_*(V)$.

(3) (a) State the Brouwer Fixed Point Theorem.
(b) Show that any $n \times n$ matrix with positive real entries has a positive real eigenvalue.