PRELIMINARY EXAMINATION IN DIFFERENTIAL TOPOLOGY
JANUARY 2015

Instructions:

- There are four questions. Attempt any three of them.
- Explain your arguments as clearly as possible.
- The exam is 90 minutes in duration.

(1) (a) Is it true that every smooth map $f\colon \mathbb{C}P^3 \to S^7$ is smoothly homotopic to a constant map?

(b) Is it true that every smooth map $f\colon S^2 \to \mathbb{C}P^3$ is smoothly homotopic to a constant map?

Justify your answers.

(2) Let $\mathsf{GL}(2n,\mathbb{R})$ denote the Lie group of invertible $2n\times 2n$-matrices, and $\mathsf{symm}(2n,\mathbb{R})$ the vector space of symmetric $2n \times 2n$-matrices. Define the symplectic group

$$\mathsf{Sp}(2n,\mathbb{R}) = \{A \in \mathsf{GL}(2n,\mathbb{R}) : A\Omega A^T = \Omega\},$$

where

$$\Omega = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}.$$

Prove that $\mathsf{Sp}(2n,\mathbb{R})$ is a submanifold of $\mathsf{GL}(2n,\mathbb{R})$ and that its tangent space at the identity matrix I is given by

$$T_I\mathsf{Sp}(2n,\mathbb{R}) = \Omega \cdot \mathsf{symm}(2n,\mathbb{R}).$$

[Hint: $(A\Omega A^T - \Omega)^T = -(A\Omega A^T - \Omega)$.]

(3) Let $\phi\colon M \to N$ be a smooth map between compact, connected, oriented manifolds (without boundary), both of dimension n. Show that for any n-form η on N, one has

$$\int_M \phi^*\eta = \mathsf{deg}(\phi) \int_N \eta,$$

where the degree $\mathsf{deg}(\phi)$ is defined in the standard way as a signed count of points in a regular level set of ϕ.

[Note: *You may quote the fact that two n-forms on N represent the same cohomology class if and only if they have the same integral.*]

(4) Let α be a 1-form on a manifold M.

(a) Show that for vector fields u and v, one has

$$(d\alpha)(u, v) = u \cdot (\alpha(v)) - v \cdot (\alpha(u)) - \alpha([u, v]).$$

[*Here d denotes the exterior derivative, $\cdot$ denotes the action of vector fields on functions f as directional derivatives, and $[u, v] \cdot f = u \cdot (v \cdot f) - v \cdot (u \cdot f)$.*]

(b) Assume α is nowhere-vanishing, and for $U \subset M$ an open set, define $\mathcal{H}(U)$ as the set of vector fields u on U such that $\alpha(u) \equiv 0$. Call α *involutive* if for every open set $U \subset M$ and every pair of vector fields $u \in \mathcal{H}(U)$ and $v \in \mathcal{H}(U)$, their bracket $[u, v]$ again lies in $\mathcal{H}(U)$.

Show that α is involutive if and only if

$$\alpha \wedge d\alpha = 0.$$