PRELIMINARY EXAMINATION IN ANALYSIS
PART I - REAL ANALYSIS
AUGUST 17, 2015, 1:00 PM - 2:30 PM

Please solve at least four of the five following problems.

(1) Let f and g be real valued measurable integrable functions on a measure space (X, μ) and let

$$F_t = \{x \in X : \ f(x) > t\}, \ G_t = \{x \in X : \ g(x) > t\}.$$

Prove that

$$\|f - g\|_1 = \int_{-\infty}^{\infty} \mu(F_t \triangle G_t) \ dt$$

where

$$F_t \triangle G_t = (F_t \setminus G_t) \cup (G_t \setminus F_t).$$

(2) Let f be a nondecreasing function on $[0, 1]$. You may assume that f is differentiable almost everywhere.

(a) Prove that

$$\int_0^1 f'(t) \ dt \leq f(1) - f(0).$$

(b) Let $\{f_n\}$ be a sequence of non-decreasing functions on $[0, 1]$ such that $F(x) = \sum_{n=1}^{\infty} f_n(x)$ converges for $x \in [a, b]$. Prove that $F'(x) = \sum_{n=1}^{\infty} f_n'(x)$ almost-everywhere.

(3) Find a non-empty closed set in $L^2([0, 1])$ which does not contain an element of minimal norm.

(4) Give an example of a sequence $\{f_h\}_{h \in \mathbb{N}} \subset L^1(\mathbb{R})$ such that $f_h \to 0$ a.e. on $\mathbb{R}$ but f_h does not converge to 0 in $L^1_{\text{loc}}(\mathbb{R})$.

(5) Let $f \in L^1(\mathbb{R})$ and φ_ε be a mollifier. This means $\varphi_\varepsilon(x) = \varepsilon^{-1}\varphi(x/\varepsilon)$ where $\varphi : \mathbb{R} \to \mathbb{R}$ is a function satisfying: $\varphi \geq 0$, the support of φ is compact and $\int \varphi = 1$. Let $f_\varepsilon = f \star \varphi_\varepsilon$ be the convolution. Show that

$$\int_{\mathbb{R}} \liminf_{\varepsilon \to 0} |f_\varepsilon| \leq \int_{\mathbb{R}} |f| \, .$$