Applied Mathematics Preliminary Exam, Part A

August 21, 2015, 10:00am – 11:30am

Work 3 of the following 4 problems.

1. Let Y be a finite dimensional subspace of a normed linear space X. Prove that Y is closed, and that there exists a continuous projection P from X onto Y. If Y is one-dimensional, describe how to construct such a projection.

2. Let X be a real Banach space with dual space X' and duality pairing $\langle .,. \rangle$. Let $A, B : X \to X'$ be linear maps.

(***a***) Assuming $\langle Ax, x \rangle \geq 0$ for all $x \in X$, show that A is bounded.
(***b***) Assuming $\langle Bx, y \rangle = \langle By, x \rangle$ for all $x, y \in X$, show that B is bounded.

3. Let $\Omega = [a, b]$ and $1 < p, q < \infty$ be given, with $\frac{1}{p} + \frac{1}{q} = 1$. Let $v \in \mathrm{L}^q(\Omega)$. For every $u \in \mathrm{L}^p(\Omega)$ define a function Au by setting

$$(Au)(t) = \int_a^t v(s)u(s)\,ds, \qquad \forall t \in \Omega.$$

(***a***) Show that A maps $\mathrm{L}^p(\Omega)$ into $\mathrm{L}^p(\Omega)$ and is continuous.
(***b***) Show that $A : \mathrm{L}^p(\Omega) \to \mathrm{L}^p(\Omega)$ is compact.

4. Let X be a complex Hilbert space with inner product $\langle .,. \rangle$, and let $A : X \to X$ be a continuous linear map that satisfies $\langle Ax, x \rangle \geq 0$ for all $x \in X$. Show that

(***a***) $\mathrm{null}(A) = [\mathrm{range}(A)]^{\perp}$.
(***b***) $\mathrm{I} + tA$ is bijective for every $t > 0$.
(***c***) $\lim_{t\to\infty} (\mathrm{I} + tA)^{-1}x = Px$ for all $x \in X$, where P is the orthogonal projection in X onto the null space $\mathrm{null}(A)$ of A.