Applied Mathematics Preliminary Exam, Part B
August 21, 2015, 11:40am – 1:10pm

Work all 3 of the following 3 problems.

1. Given $\alpha > 1$ and $\beta > 0$, consider the problem of finding a continuous function u on $\Omega = [0, 1]$ that satisfies the equation

$$u(t) = \alpha + \beta \int_0^t s \ln |u(s)| \, ds \,, \qquad \forall t \in \Omega.$$

Show that, if β is sufficiently small, then this equation possesses a unique solution $u \in U$ in some open neighborhood $U \subset \mathrm{C}(\Omega)$ of the constant function $t \mapsto \alpha$.

2. Let X and Y be normed linear spaces, and let $U \subset X$ be open. If $F : U \to Y$ is Gâteaux differentiable, and if the derivative $DF : U \to \mathcal{L}(X, Y)$ is continuous at $x \in U$, show that F is Fréchet differentiable at x.

3. Let Ω be a domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let A be a $n \times n$ matrix with components in $\mathrm{L}^\infty(\Omega)$. Let $c \in \mathrm{L}^\infty(\Omega)$ and $f \in \mathrm{L}^2(\Omega)$. Consider the boundary value problem

$$-\nabla \cdot A\nabla u + cu = f \quad \text{in } \Omega \,, \qquad\qquad u = 0 \quad \text{on } \partial\Omega \,. \tag{$\star$}$$

(All functions here are assumed to be real-valued.)
(***a***) Give the associated variational problem.

Assume now that A is symmetric and uniformly positive definite, and that c is uniformly positive. Define an energy functional $J : \mathrm{H}^1_0(\Omega) \to \mathbb{R}$ by setting

$$J(u) = \frac{1}{2} \int_\Omega \Big(\big|A^{1/2}\nabla u\big|^2 + c|u|^2 - 2fu \Big) \,, \qquad \forall u \in \mathrm{H}^1_0(\Omega) \,.$$

(***b***) Compute the derivative $DJ(u)$.
(***c***) Prove that for $u \in \mathrm{H}^1_0(\Omega)$ the following are equivalent: (i) u is a weak solution of the boundary value problem ($\star$), (ii) $DJ(u) = 0$, (iii) u minimizes J.