The University of Texas at Austin
Department of Mathematics

The Preliminary Examination in Probability
Part I

Monday, Aug 24, 2015

Problem 1. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of events on a probability space. Show that

$$\limsup_n A_n \cap \limsup_n A_n^c \subseteq \limsup_n A_n \cap A_{n+1}^c.$$

(*Note:* Remember, $\limsup_n A_n = \cap_n \cup_{k\geq n} A_k$.)

Problem 2. Let μ be a probability measure on $\mathbb{R}$, and let φ be its characteristic function. Show that μ is diffuse (has no atoms) if

$$\lim_{t\to\infty} |\varphi(t)| = \lim_{t\to-\infty} |\varphi(t)| = 0.$$

(*Hint:* For $a \in \mathbb{R}$, compute $\lim_{T\to\infty} \int_{-T}^{T} e^{-ita}\varphi(t)\,dt$.)

Problem 3. Fix a time horizon T (positive integer) and a discrete filtration $(\mathcal{F}_n)_{n=0,\dots T}$ on a fixed probability space. Let $(M_n)_{n=0,\dots T}$ be an adapted process, and assume $M_0 \in L^1$. For each $(H_n)_{n=1,\dots T}$ predictable process we define the discrete time stochastic integral

$$I(H)_n := H_1(M_1 - M_0) + \cdots + H_T(M_t - M_{T-1}), \quad n = 0, \dots T.$$

Show that, if for each bounded predictable process H we have that $I(H)_T \in L^1$ and

$$\mathbb{E}[I(H)_T] = 0,$$

then M is a martingale.