THE UNIVERSITY OF TEXAS AT AUSTIN
DEPARTMENT OF MATHEMATICS

Preliminary Examination in Probability
Part II

August 24th, 2015

Problem 2.1. Consider two pairs of adapted continuous process (H^i, X^i) defined on two filtered probability spaces $(\Omega_i, \mathcal{F}_i, (\mathcal{F}^i)_{0\leq t<\infty}, \mathbb{P}_i$ for $i = 1, 2$. Assumed that the two pairs have the same law (as two-dimensional processes), and that X^1, X^2 are semi-martingales. Show that the two stochastic integrals

$$I^i = \int H^i dX^i$$

have the same law for $i = 1, 2$.

Note: we do not really need that H^i are continuous.

Problem 2.2. Consider a binary (i.e. which takes only two values) random variable X such that $\mathbb{E}[X] = 0$. For a given Brownian motion B, construct a stopping time (with respect to its natural filtration) with the propery $\mathbb{E}[T] < \infty$ and such that B_T and X have the same distribution. Is the condition $\mathbb{E}[X] = 0$ necessary for the existence of such stopping time T?

Note: this is the simple instance of what is known as "Skorohod Imbedding".

Problem 2.3. Show that, for a continuous semimartingale M and a continous adapted process A of bounded variation, with $A_0 = 0$, we have the following equivalence

(1) M is actually a local martingale and $\langle M \rangle = A$,
(2) for each function $f \in C_b^2$ (i.e. it is C^2 and f together with its derivatives are bounded), we have that

$$f(M_t) - f(M_0) - \int_0^t f''(M_s)dA)_s$$

is a martingale.