Preliminary Examination in Topology: August 2015

Differential Topology portion

Instructions: Do all three questions.

Time Limit: 90 minutes.

1. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $F(x, y) = (xy, x^2 - 3y^2)$. Then F induces a map $f : \mathbb{RP}^1 \to \mathbb{RP}^1$.

a) Compute the Lefschetz number of f.

b) Is f homotopic to a constant map? Is f homotopic to the identity map?

c) Is the degree of f zero?

2.

a) For M a smooth manifold, with $\dim M < n$, prove that any smooth map $M \to S^n$ is smoothly homotopic to a constant map.

b) Prove that the oriented intersection number between any two closed 2-dimensional submanifolds of S^4 vanishes.

c) Prove that every smooth map $S^4 \to \mathbb{CP}^2$ has degree zero.

3.

a) Prove that every smooth 2-form α on $\mathbb{R} \times S^1$ is exact.

b) Exhibit a 2-manifold M and a smooth 2-form $\alpha \in \Omega^2(M)$ such that α is not exact; be sure to *prove* that α is not exact.

c) If α is a 1-form on a 3-manifold, do we necessarily have $\alpha \wedge \alpha = 0$? Do we necessarily have $\alpha \wedge \mathrm{d}\alpha = 0$? Give proofs or counterexamples.